Formulation of Lower Order Model for Linear Time Invariant Discrete Systems employing Mardеn Table and Genetic Algorithm

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Abstract

The model reduction is the process of approximating, as closely as possible, the dynamics of higher order linear time invariant system by a reduced order model. This paper presents a simple scheme for deriving a second order model for a given absolutely stable higher order Linear Time Invariant Discrete System (LTIDS). The proposed scheme uses a Marden Table to formulate the numerator and the denominator of an initial second order model. The coefficient of the z-term of the numerator polynomial is fixed by maintaining the transient gain ratio of the given original higher order system. The integral square error is computed by constructing the unit step responses of the original higher order system and the formulated second order model derived from Marden Table. To minimize the integral square error, so that the characteristics of the second order model closely matches the given higher order system, the genetic algorithm is applied to tune the remaining parameters of the model. The performance of the proposed method is compared with that of the other existing model reduction methods and the results are tabulated. The proposed scheme is illustrated through numerical examples.

Keywords: Linear Time Invariant Discrete Systems, Marden Table - Second order model, Unit step response, Integral square error, Genetic Algorithm.

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1. Introduction

Stability of Control Systems is important for the safe operation of the equipments that are built with components controlling their behavior. The condition of absolute stability of time invariant discrete systems can be represented in terms of the roots of the system characteristic equation to be inside the unit circle in z-plane. This can be studied using either analytical or graphical methods. The graphical methods are generally cumbersome compared with the algebraic methods and the first algebraic method is due to Schur-Cohn [1] and a simplified version is due to Marden [2]. Jury-Blanchard proposed a stability test following the Marden table in a table form [3] and further simplification on this table form is due to Raible [4] and Bistritz [5]. Several other methods are also available [6-14] for checking the absolute stability of discrete systems. Each method has its merits and applications. Aperiodicity in discrete systems is obtained when all the roots of the characteristic equation are distinct and lies on the real axis in the interval (0, 1) in the z-plane. This condition is necessary when oscillations are not desirable. In real life, an aircraft landing on a runway makes use of such control mechanisms. If these control components are not aperiodically stable, it could cause harm to the aircraft and the human life and hence critical in that perspective. E.I.Juri has developed certain schemes, namely the Bilinear Transformation [15] and the Nonlinear Transformation [16] to obtain the necessary and sufficient conditions for aperiodicity. It is highly complex to design the controllers and compensators for the stabilization of the higher order system having periodic oscillations. In this regard, a lower order model which will imitate the main characteristics of the original system may be considered for the design of controllers and compensators, leading to less computational complexity. To achieve the lower order model as discussed by various authors [1],[4-14], Marden Table which is computationally less complex and equivalent to Routh Table [17] is utilized in our proposed scheme for the formulation of initial second order model. The genetic algorithm [18] is applied to fine tune the parameters so that the second order model imitates that of the original higher order model.

2. Proposed Scheme

The proposed scheme for model reduction of linear time invariant discrete systems involving Marden Table and Genetic algorithm is as follows:

1. Let $G(z)$ be the general form of the transfer function of an nth order linear time invariant discrete system is represented as

$$G(z) = \frac{N(z)}{D(z)} = \frac{a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \ldots + a_1 z + a_0}{b_n z^n + b_{n-1}z^{n-1} + \ldots + b_1 z + b_0} \quad (1)$$

2. Consider the characteristic polynomial $D(z)$ as,

$$D(z) = b_n z^n + b_{n-1}z^{n-1} + \ldots + b_1 z + b_0 \quad (2)$$

The first step to formulate Marden table, is to reverse the coefficients of the given characteristic polynomial in equation (2), viz.

$$D_1(z) = b_0 z^n + b_1 z^{n-1} + \ldots + b_{n-1}z + b_n \quad (3)$$

Construct the Marden Table by placing $D_1(z)$ represented in equation (3) in the first row and $D(z)$ given in equation (2) in the second row. The elements in the third row are calculated as given below:

$$c_0 = b_n b_1 - b_0 b_{n-1}$$
$$c_1 = b_n b_2 - b_0 b_{n-2}$$
$$\vdots$$
$$c_n = b_n^2 - b_0^2 \quad (4)$$

and these elements form the polynomial as,

$$M_1(z): c_0 z^{n-1} + c_1 z^{n-2} + \ldots + c_{n-2}z + c_{n-1} \quad (5)$$

The coefficients of the polynomial in equation (5) is written in the reversed order as,

$$M_2(z): c_{n-1}z^{n-1} + c_{n-2}z^{n-2} + \ldots + c_1 z + c_0 \quad (6)$$

The above computations are repeated to formulate the polynomials till the constant term α_0 and this completes the formulation of the entire Marden table. The construction of Marden table is shown in Table 1.

Table 1 : Marden Table

b_0	b_1	b_2	...	b_{n-2}	b_{n-1}	b_n
b_n	b_{n-1}	b_{n-2}	...	b_2	b_1	b_0
c_0	c_1	c_2	...	c_{n-1}	c_{n-1}	
c_{n-1}	c_{n-2}	c_{n-3}	...	c_1	c_0	
d_0	d_1	d_2	...	d_{n-2}		
d_{n-2}	d_{n-3}	d_{n-4}	...	d_0		
			.			
			.			
			.			
α_0						
α_0						

Based on Table 1, Marden's stability condition for linear discrete systems having real coefficients is that, "If the constant terms b0, c0,..., α_0 computed respectively, are all positive, then the system represented by D(z) is said to be stable else it is declared as unstable". It can be observed that the even rows (2, 4 ...) of the Marden table represent the choice of coefficients of the reduced order polynomial.

3. Scale the denominator polynomial.

4. Repeat the steps 1 and 2 for the numerator polynomial, N(z).

5. For the second order model $G_2(z)$ obtained, the integral square error is calculated by comparing the unit step input responses of the original system and the obtained second order model.

6. To minimize the integral square error, Genetic algorithm (GA) is applied to adjust the coefficients of the constant term in the numerator polynomial and the 'z' and constant term in the denominator polynomial obtained from Marden table and it is used to choose the bounds for GA appropriately.

7. Repeat step 3, till the characteristics of the second order model closely matches with that of the original higher order system.

8. The integral square error is tabulated for the reduced second order model and the other models taken from the literatures in z-domain.

For the choice of the second order model, $G_2(z)$, a flow chart is shown in Fig 1.

$$G_2(z)=\frac{\alpha_1 z \pm \alpha_2}{z^2 \pm \beta_1 z \pm \beta_2} \tag{7}$$

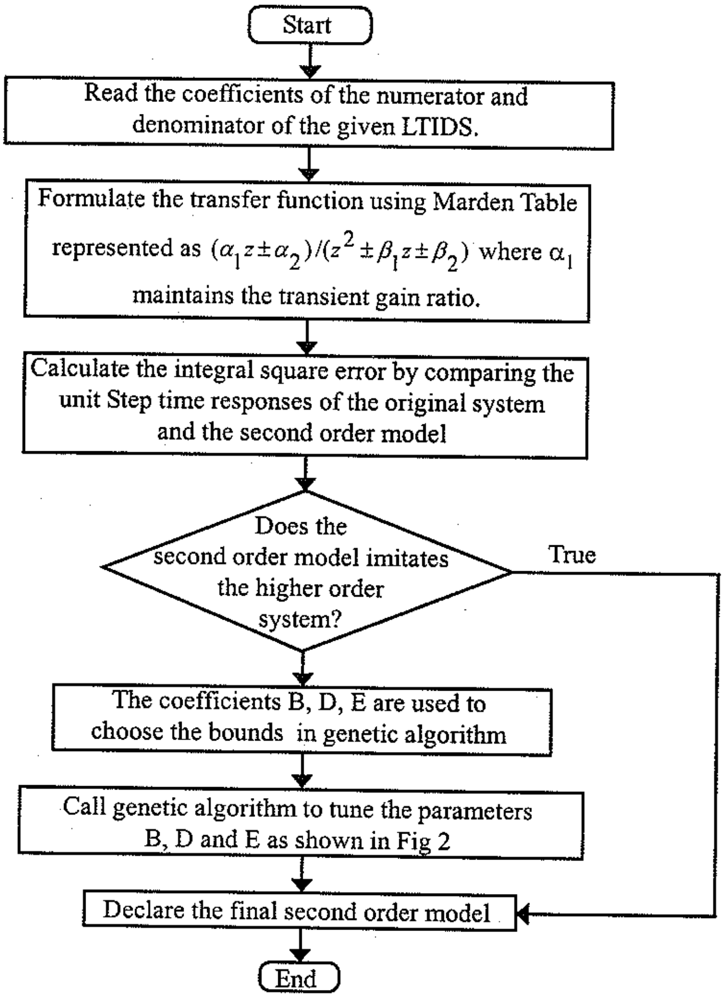

Figure 1 : Flowchart for the Formulation of the Second Order Model

The flowchart for genetic algoithm is shown in fig. 2.

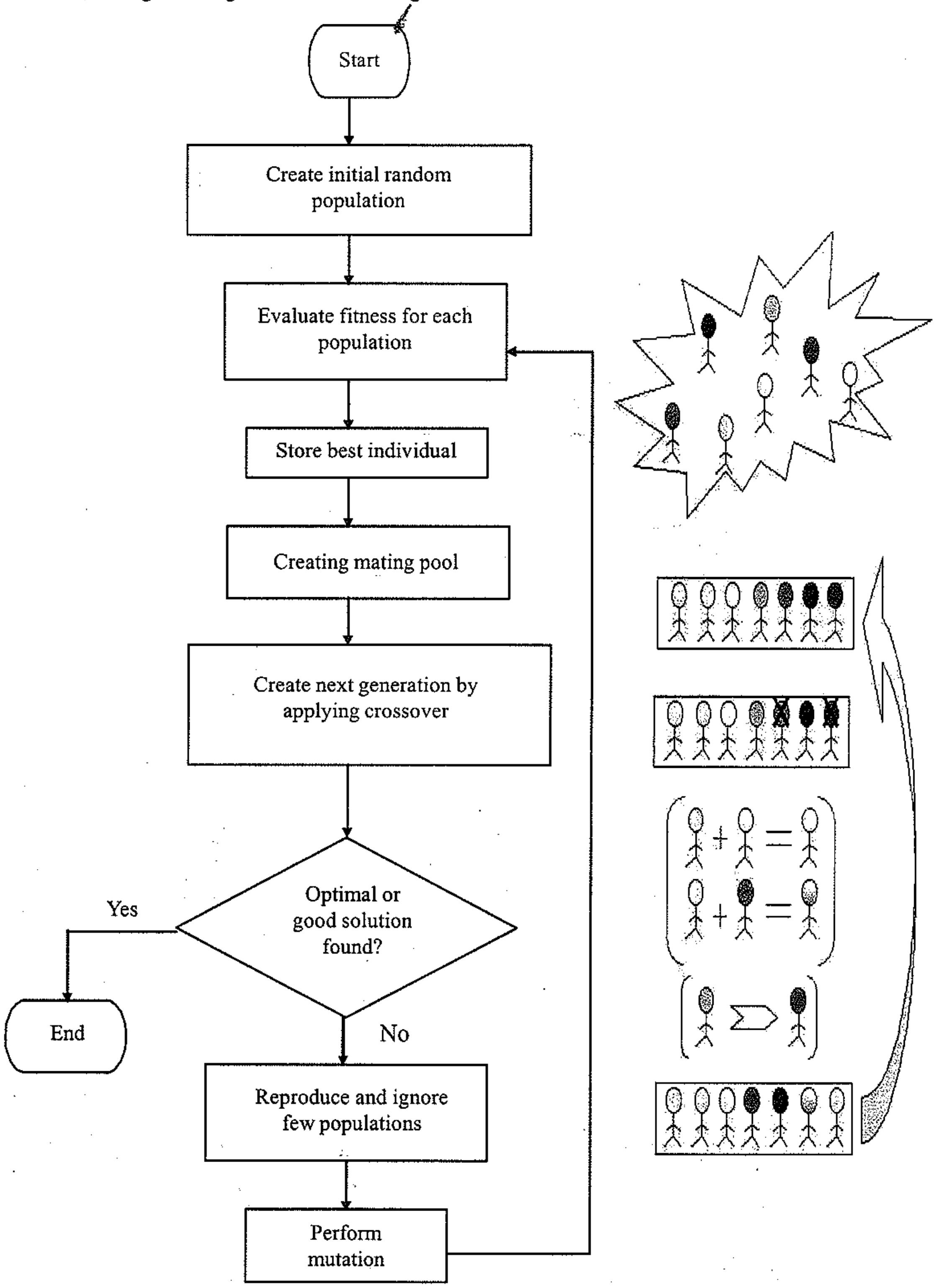

Figure 2 : Process Flow of GA

The proposed scheme is illustrated with numerical examples in the following section.

3. Numerical Examples

Example 1

1. Consider the eighth order discrete system transfer function used by Mukherjee [19]

$$G(z)=\frac{0.4209z^7+0.2793z^6-0.0526z^5+0.038z^4+ -0.1291z^3-0.0656z^2+0.011z-0.0015}{z^8-0.4209z^7-0.2793z^6+0.0526z^5-0.038z^4+ +0.1291z^3+0.0656z^2-0.011z+0.0015} \quad (1.1)$$

2. The coefficients of the various reduced order numerator polynomial of $G(z)$ obtained from the Marden Table are as follows :

0.4209	0.2793	-0.0526	0.0380	-0.1291	-0.0656	0.0110	-0.0015
0.4209	0.2793	-0.0528	0.0375	-0.1290	-0.0658	0.0120	
0.4206	0.2812	-0.0492	0.0365	-0.1275	-0.0737		
0.4076	0.2589	-0.0428	0.0278	-0.0781			
0.3926	0.2642	-0.0510	0.0775				
0.3774	0.2743	-0.1031					
0.3492	0.3492						
-0.0000							

3. The 'z' term of the numerator polynomial is derived from the original system by maintaining the same transient gain ratio thereby formulating an initial first order numerator polynomial as,

$$0.4209z - 0.0000 = 0 \quad (1.2)$$

4. The coefficients of the various reduced order denominator polynomial of $G(z)$ obtained from the Marden Table are as follows:

1.0000	-0.4209	-0.2793	0.0526	-0.0380	0.1291	0.0656	-0.0110	0.0015	1.0000
-0.4209	-0.2794	0.0524	-0.0379	0.1290	0.0660	-0.0104			
0.9999	-0.4202	-0.2781	0.0520	-0.0374	0.1261	0.0617			
0.9961	-0.4280	-0.2758	0.0488	-0.0203	0.1520				
0.9729	-0.4249	-0.2832	0.0909	0.0451					
0.9708	-0.4291	-0.2701	0.1106						
0.9582	-0.3983	-0.2212							

5. Formulate an initial second order denominator polynomial as,

$$[0.9582z^2 - 0.3983z - 0.2212] \quad (1.3)$$

6. Scale the equation (1.3) to get,

$$z^2 - 0.4157z - 0.2309 = 0 \quad (1.4)$$

7. Obtain the initial second order transfer function as

$$G_2(z) = \frac{0.4209z - 0.0000}{z^2 - 0.4157z - 0.2309} \quad (1.5)$$

8. The integral square error J is computed by comparing the unit step time responses of the original system represented by equation (1.1) and the formulated second order model represented by equation (1.5); J = 2.798225

9. Genetic algorithm is invoked to adjust the values of the parameters by maintaining the same transient gain ratio, thereby minimizing the integral square error so that the characteristics of the second order model closely matches the characteristics of the original higher order system. The final second order model $G_2(z)_f$

$$G_2(z)_f = \frac{0.4209z - 0.2104}{z^2 - 1.4787z + 0.6864},$$

having J = 0.05 (1.6)

10. The unit step responses of the original higher order discrete system given in equation (1.1) and that of the proposed second order model given in equation (1.6) is shown in Fig 3.

11. The integral square errors computed are tabulated by comparing with other models as shown in Table 2.

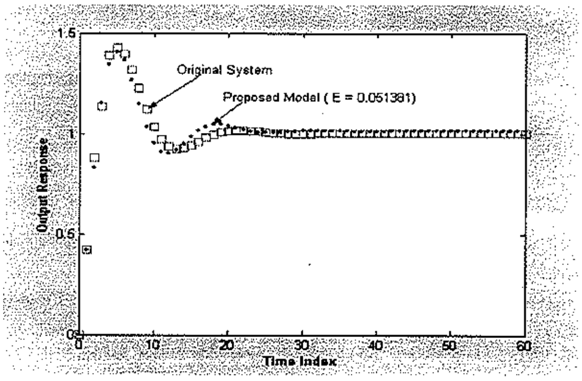

Figure 3 : Unit Step Responses for Example 1

Table 2 : Comparison of ISE of Unit Step Time Responses for Example 1

Model Reduction Method	**Reduced Model**	**Integral square error J at 60 Time Index**
Proposed model	$\frac{0.4209z - 0.2104}{z^2 - 1.4787z + 0.6864}$	0.0514
Mukherjee model [19]	$\frac{0.3973z - 0.241}{z^2 - 1.5024z + 0.6584}$	0.0986
Chen model [20]	$\frac{0.3975z - 0.318}{z^2 - 1.6025z + 0.682}$	0.3531
Shih and Wu model [21]	$\frac{0.5174z^2 + 0.0347z - 0.4825}{z^2 - 0.9303z + 0}$	1.2813

From Fig 3 and Table 2, it is observed that the proposed model yields better result when compared to the models indicated in Table 2.

Example 2

Consider the eighth order linear time invariant discrete system transfer function given by Prasad [22],

$$G(z) = \frac{1.6827z^7 + 1.116z^6 - 0.21z^5 + 0.152z^4 + -0.516z^3 - 0.262z^2 + 0.044z - 0.018}{8z^8 - 5.046z^7 - 3.348z^6 + 0.63z^5 - 0.456z^4 + +1.548z^3 + 0.786z^2 - 0.132z + 0.018} \quad (2.1)$$

2. The coefficients of the various reduced order numerator polynomial of $G(z)$ obtained from the Marden Table are as follows:

1.6827	1.1160	-0.2100	0.1520	-0.5160	-0.2620	0.0440	-0.0180
1.6825	1.1165	-0.2128	0.1465	-0.5144	-0.2642	0.0559	
1.6806	1.1253	-0.1957	0.1416	-0.5073	-0.3014		
1.6266	1.0343	-0.1703	0.1065	-0.3055			
1.5692	1.0543	-0.2023	0.3008				
1.5116	1.0931	-0.4044					
1.4034	1.3855						
0.0355							

3. The 'z' term of the numerator polynomial is derived from the original system by maintaining the same transient gain ratio thereby formulating an initial first order numerator polynomial as,

$$0.2103z + 0.0355 \quad (2.2)$$

4. The coefficients of the various reduced order denominator polynomial of $G(z)$ obtained from the Marden Table are as follows:

8.0000	-5.0460	-3.3480	0.6300	-0.4560	1.5480	0.7860	-0.1320	0.0180
8.0000	-5.0457	-3.3498	0.6265	-0.4550	1.5466	0.7935	-0.1206	
7.9981	-5.0337	-3.3264	0.6197	-0.4455	1.4961	0.7174		
7.9338	-5.1679	-3.2865	0.5641	-0.1471	1.9476			
7.4557	-5.1318	-3.4249	1.3708	1.1215				
7.2870	-5.3380	-2.9098	2.1428					
6.6569	-4.4824	-1.3401						

5. Formulate an initial second order denominator polynomial as,

$$[6.6569z^2 - 4.4824z - 1.3401] \qquad (2.3)$$

6. Scale the equation (2.3) to get,

$$z^2 - 0.6734z - 0.2013 = 0 \qquad (2.4)$$

7. Obtain the initial second order transfer function as

$$G_2(z) = \frac{0.2103z + 0.0355}{z^2 - 0.6734z - 0.2013} \qquad (2.5)$$

8. The integral square error J is computed by comparing the unit step time responses of the original system represented by equation (2.1) and the formulated second order model represented by equation (2.5), $J = 57.513005$

9. Genetic algorithm is invoked to adjust the values of the parameters by maintaining the same transient gain ratio, thereby minimizing the integral square error so that the characteristics of the second order model closely matches the characteristics of the original higher order system. The final second order model

$$G_2(z)_f = \frac{0.2103z - 0.10515}{z^2 - 1.7111z + 0.8145},$$

having $J = 0.2446$ (2.6)

10. The unit step responses of the original higher order discrete system given in equation (2.1) and that of the proposed second order model given in equation (2.6) is shown in Fig 4.

11. The integral square errors computed are tabulated by comparing with other models as shown in Table 3.

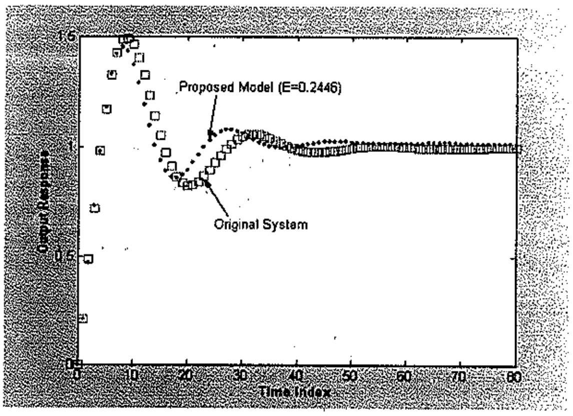

Figure 4 : Unit Step Responses for Example 2

Table 3 : Comparison of ISE of Unit Step Time Responses for Example 2

Model Reduction Method	Reduced Model	Integral square error J at 80 Time Index
Proposed model	$\frac{0.2103z - 0.10515}{z^2 - 1.7111z + 0.8145}$	0.2446
Sastry model [23]	$\frac{0.316665z - 0.277339}{z^2 - 1.683176z + 0.722779}$	0.7591
Prasad model [22]	$\frac{0.08401z - 0.03008}{z^2 - 1.73035z + 0.784282}$	2.5942

From Fig 4 and Table 3, it is observed that the integral square error of the proposed model is very less compared to other models.

EXAMPLE 3

1. Consider the system transfer function given by Mukherjee et al [24]

$$G(z) = \frac{0.4409z^3 - 0.1305z^2 + 0.0069z - 0.0047}{z^4 - 0.926423z^3 + 0.2661z^2 - 0.028z + 0.0009} \quad (3.1)$$

2. The coefficients of the various reduced order numerator polynomial of $G(z)$ obtained from the Marden Table are as follows:

0.4409	-0.1305	0.0069	-0.0047
0.4408	-0.1304	0.0055	
0.4408	-0.1288		
0.4031			

3. The 'z' term of the numerator polynomial is derived from the original system by maintaining the same transient gain ratio thereby formulating an initial first order numerator polynomial as,

$$0.4409z + 0.4031 \quad (3.2)$$

4. The coefficients of the various reduced order denominator polynomial of $G(z)$ obtained from the Marden Table are as follows:

1.0000	-0.9264	0.2661	-0.0280	0.0009
1.0000	-0.9264	0.2659	-0.0272	
0.9993-0.9192	0.2407			

5. Formulate an initial second order denominator polynomial as,

$$[0.9993z^2 - 0.9192z + 0.2407] \quad (3.3)$$

6. Scale the equation (3.3) to get,

$$z^2 - 0.9198z + 0.2409 = 0 \quad (3.4)$$

7. Obtain the initial second order transfer function as

$$G_2(z) = \frac{0.4409z + 0.4031}{z^2 - 0.9198z + 0.2409} \quad (3.5)$$

8. The integral square error J is computed by comparing the unit step time responses of the original system represented by equation (3.1) and the formulated second order model represented by equation (3.5); J = 71.687688

9. Genetic algorithm is invoked to adjust the values of the parameters by maintaining the same transient gain ratio, thereby minimizing the integral square error so that the characteristics of the second order model closely matches the characteristics of the original higher order system. The final second order model

$$G_2(z)_f = \frac{0.4409z - 0.1102}{z^2 - 0.8697z + 0.2004},$$

having J= 0.0001 (3.6)

10. The unit step responses of the original higher order discrete system given in equation (3.1) and that of the proposed second order model given in equation (3.6) is shown in Fig 5.

11. The integral square errors computed are tabulated by comparing with Mukherjee model as shown in Table 4.

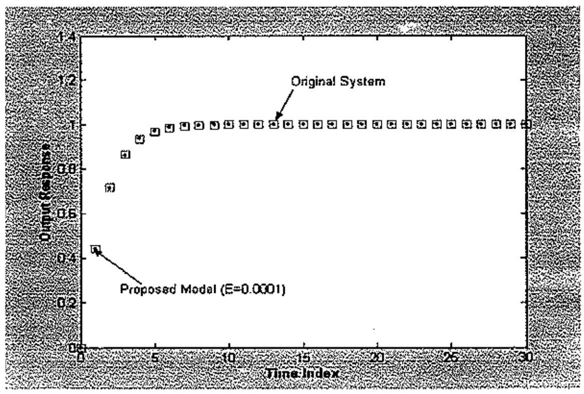

Figure 5 : Unit Step Responses for Example 3

Table 4 : Comparison of ISE of Unit Step Time Responses for Example 3

Model Reduction Method	Reduced Model	Integral square error J at 30 Time Index
Proposed model	$\frac{0.4409z-0.1102}{z^2-0.8697z+0.2004}$	0.0001
Mukherjee et al model [24]	$\frac{0.412578z+0.3495058}{z^2-0.167501z-0.0704147}$	0.0199

From Fig 5 and Table 4, it is observed that the integral square error of the proposed model is very less compared to Mukherjee model.

4. Conclusion

A new approach is proposed for deriving a second order model for a given absolutely stable linear time invariant discrete system of higher order. Using the Marden table, an initial second order model is obtained. The integral square error J is computed for the formulated second order model by comparing the unit step responses of the original higher order system and the obtained second order model. Numerical examples are illustrated with the help of the proposed scheme. Further tuning of this second order model is performed employing the GA procedure. It is observed that the proposed approach yields better results when compared to the other selected model reduction techniques from the literature [19-25]. The suggested scheme can be further used for the design of controllers and compensators [26,27].

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